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SOLUTION OF THE TWO-DIMENSIONAL INVERSE HEAT-CONDUCTION PROBLEM IN A CYLINDRICAL COORDINATE SYSTEM

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The two-dimensional inverse heat-conduction problem is considered. An algorithm of the solution and the results of a trial computation are presented.

Modern thermophysical investigation methods, thermal design, and experimental checkout of thermally stressed systems utilize the principles of inverse problems extensively, which have been recommended well in recent years. The high efficiency of methods to investigate heat-transfer processes which are based on the solution of inverse problems, especially in combination with the automated collection and processing of results, resulted in the development of inverse problems in an independent scientific aspect [1].

Different formulations of inverse heat-conduction problems (IHCP) exist at this time. Depending on the purpose, linear and nonlinear IHCP are utilized. Here one-dimensional heat-conduction models are mainly considered.

The selection of the one-dimensional models is based on those cases when a hypothesis on one-dimensional heating can be taken. This hypothesis is valid for many heat-protection

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coating structures. Thus, for instance, one-dimensional models are recommended well for the investigation of the thermal state of structures with small thicknesses and large radii of curvature. Constructive solutions affording the possibility of taking a one-dimensional heating hypothesis are attempted in the development of heat-flux sensors also. Nevertheless, the application of materials with elevated heat conductivity in modern engineering and the necessity of using design solutions resulting in the creation of heat shielding structures with small radii of curvature require taking account of a two-dimensional heat-conduction model in determining external thermal loading conditions, especially in those cases when there is no possibility of mounting special heat flux sensors utilizing the one-dimensional heat-conduction model. An example of such a structure is a cylindrical shell of small radius. It is noted in [2] that not taking account of the curvature in this case can result in errors of up to 100% and more in a computation of the temperature field. It should be noted that the nonstationary nature of the heat flux (along the generator of the thermally loaded surface) and the high heat conductivity of certain structural materials [3] in addition to the curvature of the outer surface of the structure under consideration exert influence on the two-dimensional temperature field distribution.

A two-dimensional inverse boundary-value problem of heat conduction is examined in this paper, which will permit restoration of the space-time pattern of the external thermal loading with the two-dimensional model of heat conduction taken into account. The problem is solved in a cylindrical coordinate system (Fig. 1).

Physically, such a formulation results in the following. A number of thermocouples are mounted on the inner surface of the body ($r = R_{in}$) of cylindrical shape or of the appropriate sensor of the heat-flux transducer. The heat flux delivered to the outer surface ($r = R$) is restored according to the temperatures recorded by using these thermocouples.

Let us consider an algorithm for the iteration solution of the proposed two-dimensional IHCP:

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right), \tag{1}$$

$$R_{in} < r < R, \quad 0 < \varphi < \varphi_k, \quad 0 < \tau \leqslant \tau_m, \quad T(r, \varphi, 0) = \xi(r, \varphi), \tag{2}$$

$$\frac{\partial T(r, 0, \tau)}{\partial \varphi} = \frac{\partial T(r, \varphi_k, \tau)}{\partial \varphi} = \frac{\partial T(R_{in}, \varphi, \tau)}{\partial r} = 0, \tag{3}$$

$$T(R_{in}, \varphi, \tau) = f(\varphi, \tau). \tag{4}$$

It is determined in the solution of the problem that

$$q_1(\varphi, \tau) = -\lambda \frac{\partial T(R, \varphi, \tau)}{\partial r}. \tag{5}$$

The problem (1)-(5) is solved as an optimal control problem:

$$P(\varphi, \tau) = -\frac{q(\varphi, \tau)}{\lambda}. \tag{6}$$

The control is selected from the condition of consistency of the given temperature $f(\varphi, \tau)$ with the temperature $\tilde{T}(R_{in}, \varphi, \tau)$ obtained for the control selected. The rms residual

$$I(P) = \int_0^{\tau_m} d\tau \int_0^{\varphi_k} [\tilde{T}(R_{in}, \varphi, \tau) - f(R_{in}, \varphi, \tau)]^2 d\varphi. \tag{7}$$

is considered as the measure of deviation.

In their physical substance, inverse problems are unstable. To obtain a stable IHCP solution we use the regularizing properties of the heat-conduction process and the calculational algorithm

$$P^{k+1}(\varphi, \tau) = P^k(\varphi, \tau) + \Delta P^k(\varphi, \tau), \quad k = 0, 1, \ldots, \tag{8}$$

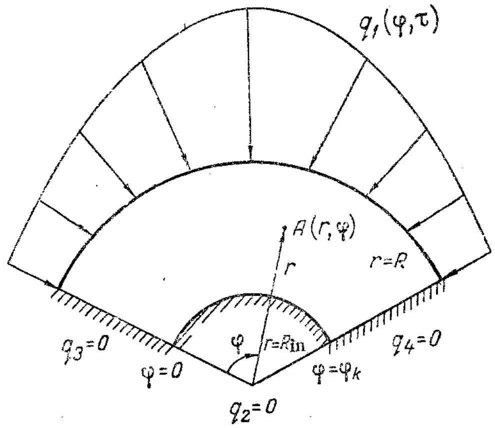

Fig. 1. Domain of the solution of a two-dimensional inverse heat-conduction problem.

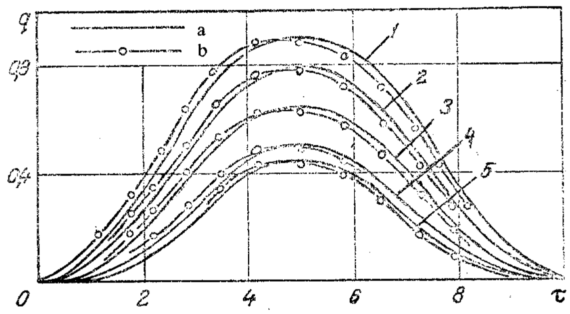

Fig. 2. Results of solving an IHCP with exact initial data: a) model values $q_1(\varphi, \tau)$; b) solutions of IHCP; 1) φ = 90; 2) 67.5; 3) 45; 4) 22.5°; 5) φ = 0). τ, sec.

where $\Delta P^k(\varphi, \tau)$ is the correction to each iteration calculated from the condition

$$I(P^{k+1}) < I(P^k). \tag{9}$$

Numerical optimization that relies on the method of conjugate gradients is used in solving the IHCP. Among the functions given on the segment $[0, \tau_m]$ the one is sought on which the functional (7) will achieve a minimal value [1].

We write the increment of the functional along $P(\varphi, \tau)$ thus:

$$\Delta I(P) \simeq 2 \int_0^{\tau_m} d\tau \int_0^{\varphi_k} \Delta T(R_{in}, \varphi, \tau)[T(R_{in}, \varphi, \tau) - f(\varphi, \tau)]\, d\varphi, \tag{10}$$

where the increment $\Delta T(R_{in}, \varphi, \tau)$ satisfies the boundary-value problem conditions

$$\frac{\partial \Delta T}{\partial \tau} = a\left(\frac{\partial^2 \Delta T}{\partial r^2} + \frac{1}{r}\frac{\partial \Delta T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Delta T}{\partial \varphi^2}\right), \tag{11}$$

$$\Delta T(r, \varphi, 0) = 0, \tag{12}$$

$$\frac{\partial \Delta T(R_{in}, \varphi, \tau)}{\partial r} = \frac{\partial \Delta T(r, 0, \tau)}{\partial \varphi} = \frac{\partial \Delta T(r, \varphi_k, \tau)}{\partial \varphi} = 0, \tag{13}$$

$$\frac{\partial \Delta T(R, \varphi, \tau)}{\partial r} = \Delta P(\varphi, \tau). \tag{14}$$

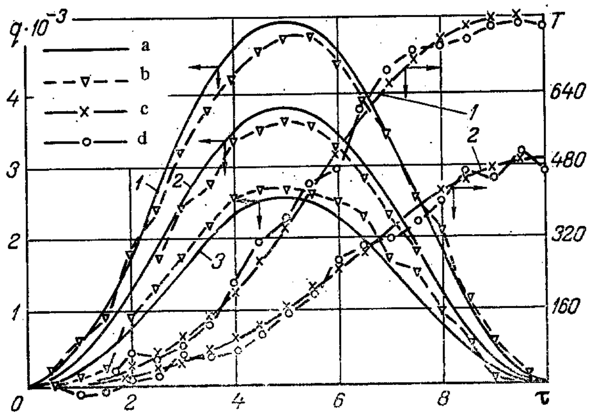

Fig. 3. Results of solving the IHCP for perturbed initial data ($\Delta Fo = 2$, the perturbation is $\delta = 5\%$ of T_{max}): a) model values $q_1(\varphi, \tau)$; b) IHCP solution; c) exact initial data; d) perturbed initial data; 1) = 90; 2) 45; 3) 0°. q in kW/m^2, T in °K, and τ, sec.

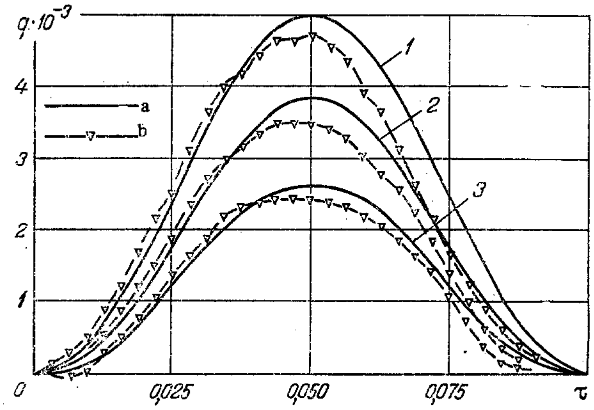

Fig. 4. Results of solving the IHCP for perturbed initial data ($\Delta Fo = 0.02$, the perturbation is $\delta = 5\%$ of T_{max}): a) model values $q_1(\varphi, \tau)$; b) IHCP solution; 1) $\varphi = 90$; 2) 45; 3) 0°.

To determine the gradient of functional $I'(P)$, we solve the conjugate problem [2]. The iteration spacing β is selected from the condition

$$\frac{\partial I(P^{k+1}(\varphi, \tau))}{\partial \beta} = 0, \tag{15}$$

where $P^{k+1}(\varphi, \tau) = P^k(\varphi, \tau) - \beta\zeta^k(\varphi, \tau)$, $\zeta^k(\varphi, \tau)$ is the direction.

Finally, we find the increment function as

$$\Delta P^{k+1} = -I'^{k+1} + \gamma^k \Delta P^k, \tag{16}$$

where $\gamma = (I'^{k+1}; I'^{k+1})/(I'^k, I'^k)$.

Consequently, to determine the next approximation, three problems must be solved: by assumption $T(r, \varphi, \tau)$, $\Delta T(r, \varphi, \tau)$ and the conjugate. They are all solved numerically.

A special investigation was performed to determine the efficiency of the algorithm. The stability of the heat-flux iteration refinement process $q(\varphi, \tau)$ and the capacity of the algorithm to restore the boundary conditions of a complex structure were verified.

Certain results of these investigations are represented in Figs. 2, 3, 4. A model for heating a copper cylindrical shell was used in the computations. It was assumed that the shell was heat insulated from the inside. The model heat flux was delivered to the outer surface. Nonstationary heat flux in both time and space (along the generator of the outer surface) was considered. The model external heat flux in sections along the space coordinate $\varphi = 0; 22.5; \ldots; 180°$ is represented in Fig. 2. The temperature field was computed for a given heat flux and the temperature was determined on the shell inner surface. The temperature obtained in this manner was perturbed by using a pseudorandom number transducer (Fig. 3) and was used as initial data for the solution of the inverse problem.

The results of investigations showed that in the case of unperturbed initial data, the inverse problem considered affords the possibility of restoring values of the nonstationary heat flux close to the model values (Fig. 2). Nearby approximations to the model heat flux were obtained also when using perturbed values of the temperature (Figs. 3 and 4) as initial data. The computations were performed here for different values of WFo — 2; 0.2; 0.02.

NOTATION

q, heat-flux density; 0, r, φ, polar coordinate system; R, R_{in}, radii of the outer and inner cylindrical shell surfaces; φ_k, greatest value of the variable φ; T, temperature; λ, a, heat-conduction and thermal diffusivity coefficients; τ, time; τ_m, greatest value of the variable τ; I, rms functional; $f(\varphi, \tau)$, temperature on the inner surface; P, a control; ΔT, a temperature increment; ζ, direction of descent; β, depth of descent; ΔFo, Fourier number spacing.

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